Noise waveforms in time domain versus frequency domain.

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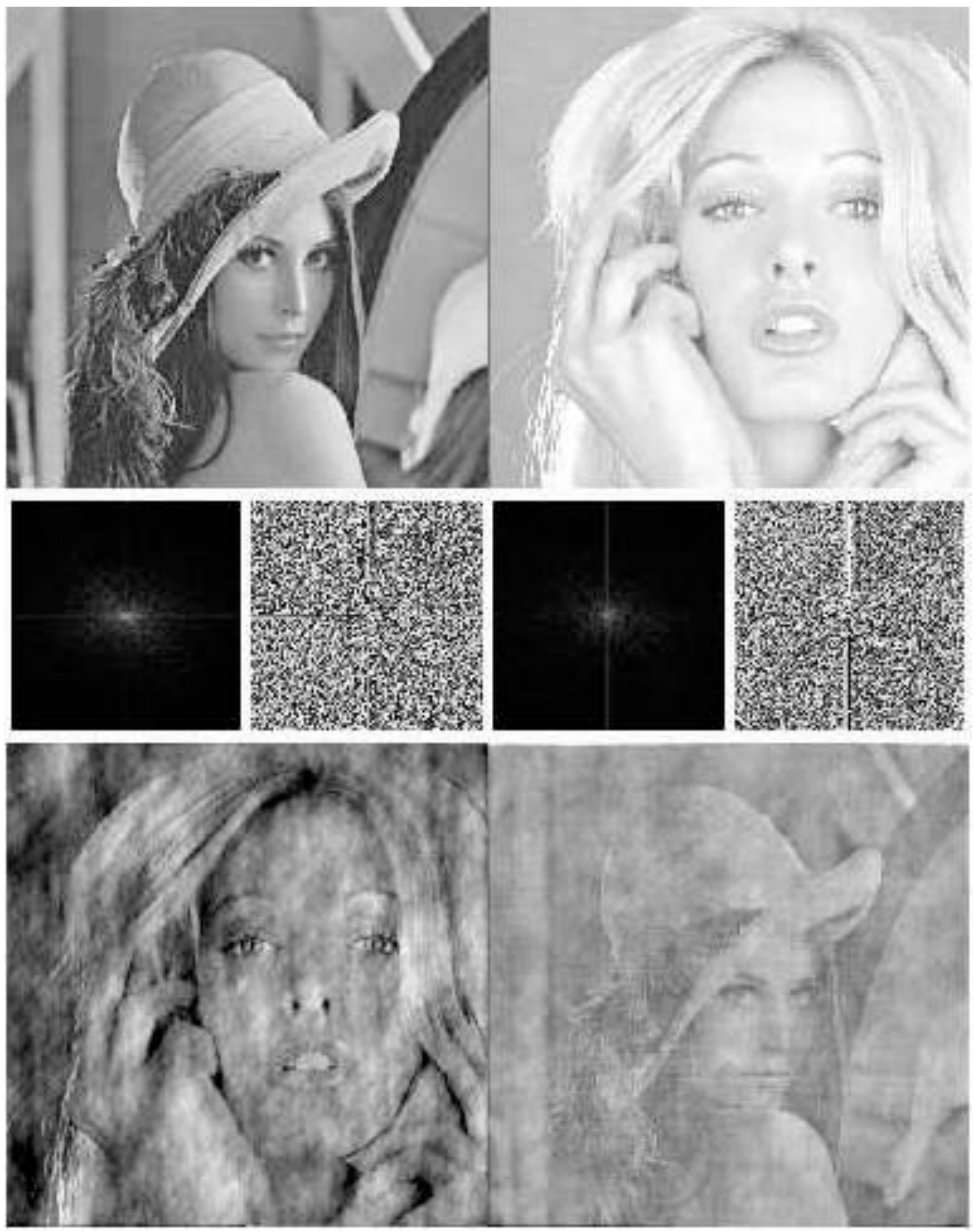
Phases in discrete Fourier transform



In the first row you see Bhanu and Gangu. Second row contains their DFS's absolute values and arguments.

In the last row there are Bhanu-Gangu (absolute values from Bhanu, arguments from Gangu) and Gangu-Bhanu (absolute values from Ganu, arguments from Bhanu).

Therefore one must be careful with both phases and magnitudes.

Commonly we tend to focus on magnitudes in analysis of signal and noise. In particular,

Question: Can noise be modeled with random phases for each frequency ?

First some simple theorems

$x_0,...,x_{N-1}$ are real numbers. Imagine it is a waveform.

$$X_l = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-i2\pi k\, l/N}$$

$$x_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} X_l e^{i2\pi k\, l/N}$$

Both of these are N-periodic: $\mathrm{X}_{l+N} = X_l$, $x_{k+N} = x_k$

using N-peridicity and that x_k are real: $X_{-l} = X_l^* = X_{N-l}$

Take the case of N to be even:

$\mathrm{X}_0 \in$ Real and $\mathrm{X}_{N/2} \in$ Real;

X_1 to $\mathrm{X}_{N/2-1}$ are complex and X_{N-1} to $\mathrm{X}_{N/2+1}$ are conjugate.

This means there are only (N/2-1)*2+2 = N independent numbers.

Take the case of N to be odd:

$\mathrm{X}_0 \in$ Real

X_1 to $\mathrm{X}_{(N-1)/2}$ are complex and X_{N-1} to $\mathrm{X}_{(N+1)/2}$ are conjugate.

This means there are only (N-1)/2*2+1=N independent numbers.

Take time interval $(0 \rightarrow T)$

In this interval we will have M samples with sample time of Δ

$$k = 0,...,M-1 \quad \tau_k = \Delta \bullet k, \quad 0 < \tau_k < T$$

We now place N elementary noise delta pulses in this interval.

q is an elementary charge with random $\pm$ sign.

$$\text{e(t)} = \sum_{i=1}^{N} q \bullet \delta(t - t_i) .... \quad \text{t}_i \text{ are random within } (0 \rightarrow T)$$

Take shaping fiter with an impulse response of $g(t) \Leftrightarrow G(\omega)$ where $\mathrm{G}(\omega)$ is defined to be the Fourier transform. Then the output noise waveform will be

$$eg(t) = \sum_{i=1}^{N} qg(t - t_i) \quad 0 < t_i < T$$

We now discretize this. We use t to denote continuous time, and τ_k for discrete times.

$$eg_k = eg(\tau_k) = \sum_{i=1}^{N} qg(\tau_k - t_i) = \sum_{i=1}^{N} qg(\Delta \cdot k - t_i)$$

We now take the DFT of the discrete waveform.

Frequency space index is l.

$\omega = l \bullet (2\pi / T) \quad l = 0, \ldots, M-1$

$$EG_l = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} eg_k \; e^{-i2\pi kl/M}$$

$$EG_l = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \left(\sum_{i=1}^{N} qg(\Delta \bullet k - t_i) \right) e^{-i2\pi kl/M}$$

$$EG_l = \frac{1}{\sqrt{M}} \sum_{i=1}^{N} \left(\sum_{k=0}^{M-1} qg(\Delta \bullet (k - k_i)) e^{-i2\pi kl/M} \right)$$

Set $t_i / \Delta = k_i$ here, but it actually does not matter.

Set $k' = k - k_i$

$$EG_l = \frac{1}{\sqrt{M}} \sum_{i=1}^{N} \left(\sum_{k'=-k_i}^{M-1-k_i} qg(\Delta \bullet k') e^{-i2\pi k' l/M} \right) e^{-i2\pi k_i l/M}$$

Recall that the Fourier transform is M-periodic, and k_i is random from 0 to M-1

$$EG_l = \frac{1}{\sqrt{M}} \sum_{i=1}^{N} e^{-i2\pi k_i l/M} \left(\sum_{k'=0}^{M-1} qg(\Delta \bullet k') e^{-i2\pi k' l/M} \right) = \sum_{i=1}^{N} q e^{-i2\pi l Ran[]} G(\omega_l)$$

There is an easier way to do the same.

$$e(t) = \sum_{i=1}^{N} q\delta(t - t_i)$$

$$E(\omega) = \sum_{i=1}^{N} q e^{-i\omega t_i}$$

Now apply the shaping filter in Fourier space

$$EG(\omega) = \sum_{i=1}^{N} q e^{-i\omega t_i} G(\omega)$$

Now we discretize ω. $\omega_l = l \bullet \dfrac{2\pi}{T}$ where $l = 0,\ldots, M$ - 1 $T = M \bullet \Delta$

$$EG_l = \sum_{i=1}^{N} q\, e^{-i(2\pi/T)\cdot l\cdot Ran[]\cdot T} G(\omega_l) = \sum_{i=1}^{N} q\, e^{-i2\pi l Ran[]} G(\omega_l)$$

Ran[] is random from 0 to 1. Must use the same random number for all l's

Conclusion: For noise due to random Poisson fluctuations, one can model the noise in frequency space with random phases multiplying the frequency spectrum (square-root of the power spectrum) using above. Time domain can be obtained by an inverse FFT.